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LETTER TO THE EDITOR

Two-index generalizations of superconformal algebras

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Abstract. The superconformal algebras of Ademollo *et al* are generalized to a multi-index form. The structure obtained is similar to the Moyal bracket analogue of the Neveu–Schwarz algebra.

1. Introduction

For some years now it has been recognized that the infinite classical Lie algebras may all be obtained from some specialization of the Moyal algebra [1–3]. The supersymmetric extension in terms of the generalization to two indices of the Neveu–Schwarz algebra is also known [1]:

$$\begin{aligned}
[L_{mn}, L_{rs}] &= \frac{1}{\sigma}\sin(\sigma(ms-nr))L_{m+r,n+s} \\
[L_{mn}, G_{rs}] &= \frac{1}{\sigma}\sin(\sigma(ms-nr))G_{m+r,n+s} \\
\{G_{mn}, G_{rs}\} &= \cos(\sigma(ms-nr))L_{m+r,n+s}\,.
\end{aligned} \tag{1}$$

The object of this letter is to discover in what form the superconformal generalizations of the Virasoro algebra of Ademollo *et al* [4] can be generalized to a multi-index algebra. First of all we shall remind the reader of the general constraints upon such an algebra, and in particular, the equivalence up to renormalization, which restrict the allowable possibilities.

The construction of a Lie algebra of the form

$$[L_{mr}, L_{ns}] = \lambda(m,r;n,s)L_{m+n,r+s} \tag{2}$$

with, obviously, the antisymmetry

$$\lambda(m,r;n,s) = -\lambda(n,s;m,r) \tag{3}$$

involves the determination of functions $\lambda(m,r;n,s)$ which fulfill the Jacobi equations

$$\sum_{\text{perm}} \lambda(m,r;n+p,s+q)\lambda(n,s;p,q) = 0 \tag{4}$$

where the sum goes over the three cyclic permutations of (m,r), (n,s) and (p,q).

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A renormalization $L^R_{m,r}$ can be performed on L_{mr} with an arbitrary function $f(m,r)$ [2]:

$$L^R_{mr} = f(m,r)L_{mr} \,. \tag{5}$$

The renormalized λ_R is

$$\lambda_R(m,r;n,s) = \lambda(m,r;n,s)R(m,r;n,s) \tag{6}$$

where the renormalization factor is

$$R(m,r;n,s) = \frac{f(m,r)f(n,s)}{f(m+n,r+s)} \,. \tag{7}$$

These renormalization factors must be taken into account to reduce a given algebra to its simplest form.

Let $\lambda^p(m,r;n,s)$ be the most general polynomial of total degree p in m,r,n,s.

$$\lambda^p(m,r;n,s) = \sum_{j=0}^{p}\sum_{k=0}^{p-j}\sum_{l=0}^{p-j-k} c^p(j,k,l,p-j-k-l)m^j r^k n^l s^{p-j-k-l} \,. \tag{8}$$

The following results were established using computer algebra.

Proposition 1

The most general solution of (4) with a finite polynomial is

$$\begin{aligned}\lambda(m,r;n,s) &= \lambda^2_S(m,r;n,s) + \lambda^1_S(m,r;n,s)\\ &= l_0(ms-nr) + l_1(m-n) + l_2(r-s)\,.\end{aligned} \tag{9}$$

In the above l_0, l_1, l_2 are arbitrary constants.

Proposition 2

The function

$$\lambda_R(m,r;n,s) = l_0\sin(\sigma(ms-nr)) \tag{10}$$

is the unique exact solution of the Jacobi equations, which extends polynomially from the quadratic case up to suitable renormalizations.

The important point is that there are no such extensions if linear terms are present in addition to the quadratic ones, in contrast to the views implicit in [3]. Suppose that we want to find a first-order correction in the parameter ϵ to, say, $\lambda^1_S(m,r;n,s)$, whose lowest degree p is of the form $\lambda^p(m,r;n,s)$. More precisely, suppose

$$\lambda(m,r;n,s) = \lambda^1_S(m,r;n,s) + \epsilon(\lambda^p(m,r;n,s) + \cdots) \tag{11}$$

where $\cdots$ stands for terms of degree higher than p. Then defining the generic functions $E(q,p)$ with which the Jacobi identities can be expressed as

$$E(q,p) = \sum_{\text{perm}}(\lambda^q(m,r;n+k,s+t)\lambda^p(n,s;k,t) + \lambda^p(m,r;n+k,s+t)\lambda^q(n,s;k,t)) \tag{12}$$

it is easy to show that, to first non-trivial order in ϵ, $\lambda^p(m,r;n,s)$ has to satisfy

$$E(1,p) = 0\,. \tag{13}$$

Using a renormalization (see equation (6)) with $f(m,r)$ of the form

$$f(m,r) = 1 - \epsilon g^{p-1}(m,r) \tag{14}$$

with $g^{p-1}(m, r)$ an arbitrary function of m and s of total degree $p-1$, one can easily see that the correction in ϵ in (13) is renormalized to

$$\epsilon\lambda_R^p(m, r; n, s) = \epsilon\left(\lambda^p((m, r; n, s) - h^{p-1}(m, r; n, s)\lambda_S^1((m, r; n, s)\right) \tag{15}$$

where

$$h^{p-1}(m, r; n, s) = g^{p-1}(m, r) + g^{p-1}(n, s) - g^{p-1}(m+n, r+s)\,. \tag{16}$$

The results can be summarized as follows.

Proposition 3

Except for $p = 2$, the most general correction of the form (15) to the linear exact solution can be renormalized away to $\epsilon\lambda_R^p(m, r;\,) = 0$ by choosing $g^{p-1}(m, r; n, s)$ suitably. This has been checked explicitly for $3 \leqslant p \leqslant 7$. It follows that the conjecture that 'there does not exist any analytical λ whose first order is linear in its variables' appears extremely likely.

2. The generalization to arbitrary superconformal algebras

First of all we obtain the most general algebra corresponding to the conformal superalgebra found by Ademollo *et al* in the case $N = 2$ under the hypothesis that the structure functions are polynomials of degree at most two in their variables. This algebra takes the following form, dependent upon one parameter p found using REDUCE:

$$\begin{aligned}
&[L_{mr}, L_{ns}] = ((ms - nr) + p(m-n))L_{m+n,r+s}\\
&[L_{mr}, T_{ns}] = ((ms - nr) - pn)T_{m+n,r+s}\\
&[L_{mr}, G^\alpha_{ns}] = ((ms - nr) + \tfrac{1}{2}p(m-2n))G^\alpha_{m+n,r+s}\\
&[T_{mr}, T_{ns}] = 0\\
&[T_{mr}, G^\alpha_{ns}] = \epsilon^{\alpha\beta}G^\beta_{m+n,r+s}\\
&\{G^\alpha_{mr}, G^\beta_{ns}\} = \delta^{\alpha\beta}L_{m+n,r+s} - \epsilon^{\alpha\beta}((ms - nr) + \tfrac{1}{2}p(m-n))T_{m+n,r+s}\,.
\end{aligned} \tag{17}$$

Here $\epsilon^{\alpha\beta}$ is the usual two-index antisymmetric symbol. An equivalent algebra, without the free parameter, has appeared recently in a preprint by Buffon *et al* [5].

The solution corresponding to the Moyal algebra, as found by calculations using REDUCE, is as follows:

$$\begin{aligned}
&[L_{mr}, L_{ns}] = \sin(ms - nr)L_{m+n,r+s}\\
&[L_{mr}, T_{ns}] = \sin(ms - nr)T_{m+n,r+s}\\
&[L_{mr}, G^\alpha_{ns}] = \sin(ms - nr)G^\alpha_{m+n,r+s}\\
&[T_{mr}, T_{ns}] = \sin(ms - nr)L_{m+n,r+s}\\
&[T_{mr}, G^\alpha_{ns}] = \epsilon^{\alpha\beta}\cos(ms - nr)G^\beta_{m+n,r+s}\\
&\{G^\alpha_{mr}, G^\beta_{ns}\} = \delta^{\alpha\beta}\cos(ms - nr)L_{m+n,r+s} - \epsilon^{\alpha\beta}\sin(ms - nr)T_{m+n,r+s}\,.
\end{aligned} \tag{18}$$

An inessential parameter σ (as in equations (1)) has been set to unity.

3. Extension to *N* = 4

The $N = 4$ case (the greek indices below run over the values from 1 to 4) may be treated in a similar fashion. With $L,\ U,\ G^\mu,\ Q^\mu$ and $A^{\mu\nu}$ behaving as a scalar, pseudoscalar, vector, axial vector and tensor respectively under the action of the orthogonal group, a solution of the 86 identities which are a consequence of the 35 Jacobi identities of the problem were found, using REDUCE. Introducing the abbreviations

$$S = \sin(\boldsymbol{m} \wedge \boldsymbol{n})$$

$$C = \cos(\boldsymbol{m} \wedge \boldsymbol{n}) \tag{19}$$

where $\boldsymbol{m},\ \boldsymbol{n}$ are real $2k$-dimensional (or $(2k+1)$-dimensional) vectors and $\boldsymbol{m} \wedge \boldsymbol{n}$ signifies

$$\boldsymbol{m} \wedge \boldsymbol{n} = \sigma_{1,2}(m_1 n_2 - m_2 n_1) + \cdots + \sigma_{2k-1,2k}(m_{2k-1} n_{2k} - m_{2k} n_{2k-1}) \tag{20}$$

with the coefficients $\sigma_{i,i+1}$ being treated as constant parameters, one finds the following algebra:

$$\begin{aligned}
&[L_{\boldsymbol{m}}, L_{\boldsymbol{n}}] = S L_{\boldsymbol{m}+\boldsymbol{n}}\\
&[L_{\boldsymbol{m}}, G^\mu_{\boldsymbol{n}}] = S G^\mu_{\boldsymbol{m}+\boldsymbol{n}}\\
&[L_{\boldsymbol{m}}, A^{\mu\nu}_{\boldsymbol{n}}] = S A^{\mu\nu}_{\boldsymbol{m}+\boldsymbol{n}}\\
&[L_{\boldsymbol{m}}, Q^\mu_{\boldsymbol{n}}] = S Q^\mu_{\boldsymbol{m}+\boldsymbol{n}}\\
&[L_{\boldsymbol{m}}, U_{\boldsymbol{n}}] = S U_{\boldsymbol{m}+\boldsymbol{n}}\\
&\{G^\mu_{\boldsymbol{m}}, G^\nu_{\boldsymbol{n}}\} = C\delta^{\mu\nu} L_{\boldsymbol{m}+\boldsymbol{n}} + S A^{\mu\nu}_{\boldsymbol{m}+\boldsymbol{n}}\\
&[G^\mu_{\boldsymbol{m}}, A^{\nu\rho}_{\boldsymbol{n}}] = C(\delta^{\mu\nu} G^\rho_{\boldsymbol{m}+\boldsymbol{n}} - \delta^{\mu\rho} G^\nu_{\boldsymbol{m}+\boldsymbol{n}}) + S\epsilon^{\mu\nu\rho\sigma} Q^\sigma_{\boldsymbol{m}+\boldsymbol{n}}\\
&\{G^\mu_{\boldsymbol{m}}, Q^\nu_{\boldsymbol{n}}\} = -S\delta^{\mu\nu} U_{\boldsymbol{m}+\boldsymbol{n}} + \tfrac{1}{2} C\epsilon^{\mu\nu\rho\sigma} A^{\rho\sigma}_{\boldsymbol{m}+\boldsymbol{n}}\\
&[G^\mu_{\boldsymbol{m}}, U_{\boldsymbol{n}}] = -C Q^\mu_{\boldsymbol{m}+\boldsymbol{n}} &(21)\\
&[A^{\mu\nu}_{\boldsymbol{m}}, A^{\rho\sigma}_{\boldsymbol{n}}] = -S(\delta^{\mu\sigma}\delta^{\nu\rho} - \delta^{\mu\rho}\delta^{\nu\sigma}) L_{\boldsymbol{m}+\boldsymbol{n}} + C\Big(\delta^{\mu\sigma} A^{\nu\rho}_{\boldsymbol{m}+\boldsymbol{n}} + \delta^{\nu\rho} A^{\mu\sigma}_{\boldsymbol{m}+\boldsymbol{n}}\\
&\qquad\qquad -\delta^{\mu\rho} A^{\nu\sigma}_{\boldsymbol{m}+\boldsymbol{n}} - \delta^{\nu\sigma} A^{\mu\rho}_{\boldsymbol{m}+\boldsymbol{n}}\Big) + S\epsilon^{\mu\nu\rho\sigma} U_{\boldsymbol{m}+\boldsymbol{n}}\\
&[A^{\mu\nu}_{\boldsymbol{m}}, Q^\rho_{\boldsymbol{n}}] = -S\epsilon^{\mu\nu\rho\sigma} G^\sigma_{\boldsymbol{m}+\boldsymbol{n}} + C\Big(\delta^{\mu\rho} Q^\nu_{\boldsymbol{m}+\boldsymbol{n}} - \delta^{\nu\rho} Q^\mu_{\boldsymbol{m}+\boldsymbol{n}}\Big)\\
&[A^{\mu\nu}_{\boldsymbol{m}}, U_{\boldsymbol{n}}] = \tfrac{1}{2} S\epsilon^{\mu\nu\rho\sigma} A^{\rho\sigma}_{\boldsymbol{m}+\boldsymbol{n}}\\
&\{Q^\mu_{\boldsymbol{m}}, Q^\nu_{\boldsymbol{n}}\} = C\delta^{\mu\nu} L_{\boldsymbol{m}+\boldsymbol{n}} + S A^{\mu\nu}_{\boldsymbol{m}+\boldsymbol{n}}\\
&[Q^\mu_{\boldsymbol{m}}, U_{\boldsymbol{n}}] = C G^\mu_{\boldsymbol{m}+\boldsymbol{n}}\\
&[U\boldsymbol{m}, U_{\boldsymbol{n}}] = S L_{\boldsymbol{m}+\boldsymbol{n}}\,.
\end{aligned}$$

In the anticommutator $\{G, Q\}$ and the commutators $[G, U]$ and $[Q, U]$, one might have expected C where S appears and vice versa, from the permutation symmetry analogy with the one-index $N = 4$ Ademollo algebra.

However, there is a representation of this algebra (for two-vector indices) in terms of functions $F(x, y)_{mn}$ which act as basis functions for the star product, tensored with a Clifford algebra. The motivation which lies behind this representation comes from the precise behaviour of the generators under the orthogonal group. This procedure obviously generalizes to even N. The star product of two functions $F(x, y)$ and $G(x, y)$ is the associative product

$$F(x, y) \star G(x, y) = \lim_{\substack{x' \to x \\ y' \to y}} \exp\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y'} - \frac{\partial}{\partial x'}\frac{\partial}{\partial y}\right) F(x, y) G(x', y') \,. \qquad (22)$$

Choose $F(x, y)_{mn} = \exp(mx + \mathrm{i}ny)$ and represent the operators as follows, in terms of the a direct product of the familiar four-dimensional Dirac matrices γ^μ, $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ and the functions F_{mn}:

$$\begin{aligned} L_{mn} &= \mathbb{1} \otimes F_{mn} \\ G^\mu_{mn} &= \gamma^\mu \otimes F_{mn} \\ A^{\mu,\nu}_{mn} &= \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \otimes F_{mn} \\ Q^\mu_{mn} &= \gamma^5\gamma^\mu \otimes F_{mn} \\ U_{mn} &= \gamma^5 \otimes F_{mn} \,. \end{aligned} \qquad (23)$$

These assignments provide a representation of the algebra (21) where the product is used in the sense of the star product. In the case where $\lambda = 2\pi/\text{integer}$, $F_{m,n}$ can be represented by matrices in the Weyl representation of a unitary algebra, and the product is then ordinary multiplication [1].

4. Conclusion

It is found that the generalization of infinite superconformal algebras to multi-index algebras is of a surprisingly restrictive type, being patterned after the Neveu–Schwarz example as a direct product of a Clifford algebra with a star product.

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Note added. As this letter went to press, we were informed by C K Zachos that algebras similar to (17)–(18) also make an appearance in the paper of Fradkin E S and Linetsky I Ya 1991 *Mod. Phys. Lett.* **6A** 217–24

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